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Solving the Helmholtz Equation for the Neumann Boundary Condition for the Pseudosphere by the Galerkin Method

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Abstract

In this paper, the Helmholtz equation for the exterior Neumann boundary condition for the pseudosphere in three dimensions using the global Galerkin method is studied. The Galerkin method will be used to solve Jones' modified integral equation approach (modified as a series of radiating waves will be added to the fundamental solution) for the Neumann problem for the Helmholtz equation, which uses a series of double sums to approximate the integral. A Fortran 77 program is used and some required subroutines from the Naval Warfare Center are called to help increase our accuracy since these boundary integrals are difficult to solve. The solutions obtained are compared to the true solution for the Neumann problem to understand how well the method converges. The lower errors obtained show that the method for complete reflection of the sound waves off of the pseudosphere is accurate and successful. Also presented in this paper are both computational and theoretical details of the method of different values of k for the pseudosphere.

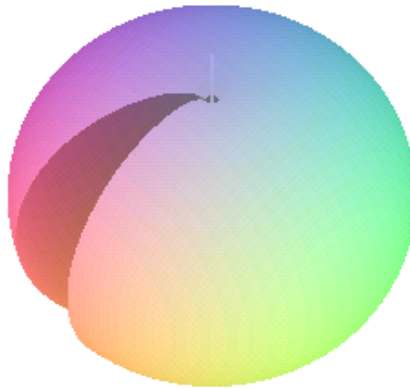
Introduction

The Helmholtz equation is used in many instances and can be applied to different problems, mostly involving different types of wave formulations. It is commonly used for many types of radiation and sound problems and all attempt to find a solution to it. In our case, we studied the Helmholtz equation with the Neumann boundary condition for the pseudosphere in 3 dimensions using the Galerkin Method. The equation for the pseudosphere is given by

$$x = \cos(x) \sin(y), \quad y = \sin(x) \cos(y), \quad z = \cos(y) + c[\log(\tan \frac{y}{2})]. \quad (1)$$

Boris Grigoryevich Galerkin, a Russian mathematician, developed the Galerkin method using weighted residual models. Galerkin's method selects weight functions from basis functions, such as $w(x) \in \{\phi_i(x)\}_{i=1}^n$. To apply the Galerkin method, the coefficients $\{\phi_i\}_{i=1}^n$ are solved for in the equation, $\int_a^b \phi_i(x)(L[u(x)] + f(x))dx = 0$ for $i = 1, 2, \dots, n$. Using boundary conditions, complex integral equations can be approximated. In Criado and Alamo's paper, pseudospheres are discussed as a natural occurrence of a hyperbolic function [1]. In the case of Thomas rotation, the paper also describes how the surface is a pseudosphere corresponding to the space of relativistic velocities [1]. An application of the Helmholtz equation is how tidal waves are scattered amongst reefs and spits. The same methods of solving the Helmholtz equation apply where waves are scattering from are drastically different surfaces. Other areas of physics explore the use of the Helmholtz equation such as the study of electromagnetic radiation and seismology, but largely the equation is used in the study of acoustics. Some shapes have already been explored using the Helmholtz equation and its uses with acoustics. The shapes researched and published have been the sphere, ellipsoid, the Oval of Cassini, which is the shape of a peanut and the perturbation of the sphere. The exploration of the pseudosphere uses the same type of programming, experimentation and approximation, but slightly altered because the shape has a singularity at its poles and that creates problems in terms of differentiability of the boundary.

Figure 1: The pseudosphere



We altered the coefficients c so that the pseudosphere becomes a somewhat smooth, integrable curve near its poles. We used c values of 1, 0.01, 0.001, and 0.000001. The pseudosphere is most commonly recognized and used in hyperbolic geometry and often refers to a specific surface known as the tractricoid. This surface results as revolving a curve about its asymptote. Thus, the term pseudosphere, with a radius of r , refers to any surface of curvature $\frac{-1}{r^2}$.

Figure 2: Cross sectional view of the pseudosphere when $c = 0.07$

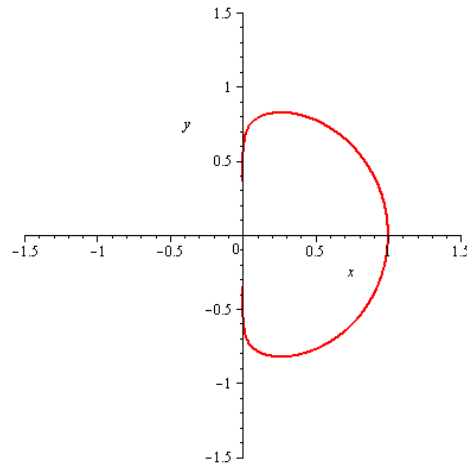
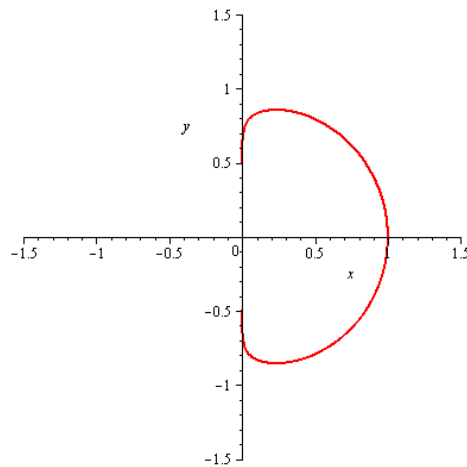


Figure 3: Cross sectional view of the pseudosphere when $c = 0.055$



Theorems & Definitions

- The Hankel functions of the 1st kind are a solution of Bessel's differential equation and are used to represent outward-propagator cylindrical wave solutions of the cylindrical wave equation.
- Spherical harmonics are defined as the angular portion of a set of solutions to the Laplace equation. Using a set of spherical coordinates, Laplace's spherical harmonics form a specific set of spherical harmonics and they then make up an orthogonal system. Spherical harmonics are commonly used in the study of magnetic field of planets and stars, and also in 3D computer graphics, including playing a role in false illumination and the radiosity equation.
- Uniformly convergent- the entire function converges at the same rate; not just at a comparable pointwise convergence, but at a specific point of a function.
- Holder continuity - $|f(x) - f(y)| \leq C|x - y|^\alpha$ where constants C and α are nonnegative and real.
- Compact Set - A subset S of a topological space X is compact if for every open cover of S there exists a finite subcover of S .
- Bounded Set - A set S in a metric space (S, d) is bounded if it has a finite generalized diameter, i.e., there is an $R < \infty$ such that $d(x, y) \leq R$ for all $x, y \in S$. A set in R^n is bounded if it is contained inside some ball $x_1^2 + \dots + x_n^2 \leq R^2$ of finite radius R .
- Kernel - in integral calculus, a function of two variables used to define a mapping to the zero element. It is also known as the integral kernel or kernel function. An example of such is the Green's function. In this case, our kernel is defined as, $\frac{e^{ikr}}{r} + \chi(x, y)$, where the first term is the fundamental solution of the Helmholtz Equation and the second term is the infinite series.
- Green's Theorem states that the value of a double integral over a simple connected plane region R is determined by the value of a line integral around the boundary of R . When applied to a plane, we assume that R is a piecewise smooth simple closed curve within a region R . If P , Q , $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on R , then, $\int_C Pdx + Qdy = \int_R [(\frac{\partial Q}{\partial x}) - (\frac{\partial P}{\partial y})]dA$.
- The inner product of two functions, f_1 and f_2 on an interval $[a, b]$ is the number $(f_1, f_2) = \int_b^a f_1(x)f_2(x)dx$.

Method

The research we are performing is using the Helmholtz equation and using the Galerkin Numerical method to solve for the way that sound waves are completely reflected from the pseudosphere shape. By altering the constants in front of the z term of this shape, we can manipulate its depth to have a shape that is smoother and continuous, which in turn is easier to integrate. The general form of this equation is given by,

$$\Delta u + k^2 u = 0, \text{Im } k \geq 0 \tag{2}$$

where k is the wave number.

Also, $u(p)$ = partial derivative of the given function $f(p)$, for any point p on the boundary of the pseudosphere and is given by the equation,

$$f(p) = \frac{\partial u(p)}{\partial \nu_p}, \quad p \in S. \quad (3)$$

Using certain boundary conditions, in our case the Neumann condition for this integral equation, we hope to find an approximate solution to exterior boundary value problem. As we will use Green's Theorem, we solve the Helmholtz equation only on the boundary of the pseudosphere. The fundamental solution of the Helmholtz equation is given by,

$$u_1(x, y, z) = \frac{e^{ikr}}{r}. \quad (4)$$

As there is a singularity when $r = 0$, as ($r = |p - q|$), we will add an infinite series of radiating waves to our integral equation. The series of radiating waves is given by,

$$\chi(A, q) = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} a_{nm} h_n^{(1)}(k|A|) Y_n^m\left(\frac{A}{|A|}\right) h_n^{(1)}(k|q|) \bar{Y}_n^m\left(\frac{q}{|q|}\right). \quad (5)$$

In regards to the term $h_n^{(1)}$, Kleinman and Roach gave an optimal coefficient for the Dirichlet problem, with the radius of a sphere, R , centered at the origin [4]. The upper bound of the spectral radius is minimized by this explicit form of the term. In our case, R is the radius of our pseudosphere. We have a coefficient, a_{nm} , for spherical regions, however our shape is a hyperbolic one. The equation for a_{nm} is given by,

$$a_{nm} = -\frac{1}{2} \left(\frac{j_n(kR)}{h_n^{(1)}(kR)} + \frac{j_n'(kR)}{h_n^{(1)'}(kR)} \right) \quad (6)$$

for $n = 0, 1, 2, \dots$, and $m = -n, \dots, n$.

We are going to use the same coefficient for our pseudosphere, because we will minimize the poles of the shape, by choosing c to be quite small, reducing its "hyperbolicness," further reducing the error. As a counterexample, we will run our program with a c that manipulates the pseudosphere to become extremely hyperbolic and show how the resulting integral does not converge. In conclusion, the more long and defined the poles of our shape are, the more hyperbolic it becomes raising our error.

The concept of linearity is an important theme in integral equations. Linearity is applied to functions and integral equations; it has the following properties,

$$(f + g)(x) = f(x) + g(x) \quad (7)$$

$$(\alpha f)(x) = \alpha f(x). \quad (8)$$

Linear operators are a mapping, T , from a vector space, X , to a vector space, Y , are said to be linear if it satisfies the equation,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y). \quad (9)$$

The integral operator, also called the kernel operator, will be the most useful and important linear operator for our research and is defined as,

$$(Kf)(x) \equiv \int_0^1 K(x, y)f(y)dy. \quad (10)$$

Where $K(x, y)$ is the kernel. As previously noted, our kernel is defined as, $\frac{e^{ikr}}{r} + \chi(x, y)$. Below, the integral operator is proved to be linear.

Proposition 1: The integral operator is linear.

Proof: We wish to show that $[K(f + g)](x) = (Kf)(x) + (Kg)(x)$.

Then, $[K(f + g)](x) = \int_0^1 K(x, y)[f + g](y)dy$.

Next, we distribute (y) , $= \int_0^1 K(x, y)[f(y) + g(y)]dy$.

Separate the integrals, $= \int_0^1 K(x, y)f(y)dy + \int_0^1 K(x, y)g(y)dy$.

$= (Kf)(x) + (Kg)(x) \quad \square$

Proof: We now wish to show that $K(\alpha f(x)) = (\alpha K)f(x)$.

$K(\alpha f) = \int_0^1 K(x, y)(\alpha f)(y)dy$

$= \int_0^1 K(x, y)\alpha f(y)dy$

$= \int_0^1 \alpha K(x, y)f(y)dy$

$= (\alpha K)(f(x))$

The integral operator is linear \square

Proposition 2: K^2 is the composition of two integrals, resulting in a double integral.

Proof: $(K^2 f)(x) = [K(Kf)(x)]$

$= \int_0^1 K(x, y)[Kf](y)dy$

$= \int_0^1 K(x, y) \int_0^1 K(y, z)f(z)dzdy$

$= \int_0^1 \int_0^1 K(x, y)K(y, z)f(z)dzdy \quad \square$

In working with integral equations, we encounter function spaces. The maximum number of linearly independent vectors are referred to as the basis of a vector space. A norm is one way to define a vector

space. A norm is intended to capture the essence of size of an abstract space and has the following properties,

$$(i) \|x\| > 0, x \neq 0 \tag{11}$$

$$(ii) \|\alpha x\| = |\alpha| \|x\| \tag{12}$$

$$(iii) \|x + y\| \leq \|x\| + \|y\|. \tag{13}$$

An important point to note is that condition three is the important Triangle Inequality. Examples of norms include the Euclidean norm, referred to as also L_2 and $\|x\|_2$, which equals $[\sum_{i=1}^n x_i^2]^{\frac{1}{2}}$ and the uniform, or infinite norm, $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$. Lastly, the p Norm is defined as, $\|x\|_p = [\sum_{i=1}^n x_i^p]^{\frac{1}{p}}$.

Numerical Methods: Galerkin Method

The main task of numerical methods is to make the transition in such a way that we reliably capture all relevant features of the original function. To do so, we use the Galerkin Method to solve the Modified (we say modified as we are adding a series of radiating waves to the fundamental solution) Neumann problem for the Helmholtz equation, which uses a series of double sums to approximate the integral. The integral is approximated by,

$$I_m(f) = \frac{\pi}{M} \sum_{i=1}^{2M} \sum_{j=1}^{2M} w_j f(\phi_i, \theta_j). \tag{14}$$

The Galerkin Method is defined on a Hilbert space. A vector space that has a norm is called a normed linear space. A complete normed linear space is called a Banach space. Completeness guarantees that there is always an x in the space to converge to $\lim_{n \rightarrow \infty} \|x - x_n\| = 0$. A Banach space whose norm comes from an inner product is defined as a Hilbert space. Euclidean n space is called a Hilbert space with the inner product corresponding to the standard dot product, $[\int f(x)g(x)dx]^{\frac{1}{2}}$. The 1-norm and infinite norm are not Hilbert spaces.

We will use a Fortran 77 program and will call some required subroutines from the Naval Warfare Center to help increase our accuracy because these boundary integrals are incredibly difficult to solve. We'll use the solutions found and compare them to the true solution from the Neumann Problem to see how well the method converges. The lower the error, it follows that the problem has worked out very well for complete reflection of the sound waves off of the pseudosphere.

Numerical Results

Below the data that has been collected is shown. We can change the wave number, k , the number of interior nodes, NINTI, the number of exterior nodes, NINTE and the values of the constant, c , to obtain better convergence results. NINTE must be less than NINTI because the outer integral is smoother than the inner integrals.

TABLE 1 $k = 1, c = 0.001, NDEG = 7, NINTI = 32, NINTE = 16$, true solution u_1

| <i>point</i> | <i>absolute error</i> |
|--------------|------------------------|
| (20, 21, 22) | 3.646×10^{-6} |
| (10, 11, 12) | 3.748×10^{-6} |
| (12, 4, 11) | 2.537×10^{-6} |
| (5, 6, 7) | 6.46×10^{-6} |
| (2, 3, 4) | 5.987×10^{-5} |

TABLE 2 $k = 1, c = 0.01, NDEG = 7, NINTI = 32, NINTE = 16$, true solution u_1

| <i>point</i> | <i>absolute error</i> |
|--------------|------------------------|
| (20, 21, 22) | 7.931×10^{-6} |
| (10, 11, 12) | 3.934×10^{-6} |
| (12, 4, 11) | 2.712×10^{-5} |
| (5, 6, 7) | 6.194×10^{-5} |
| (2, 3, 4) | 3.176×10^{-4} |

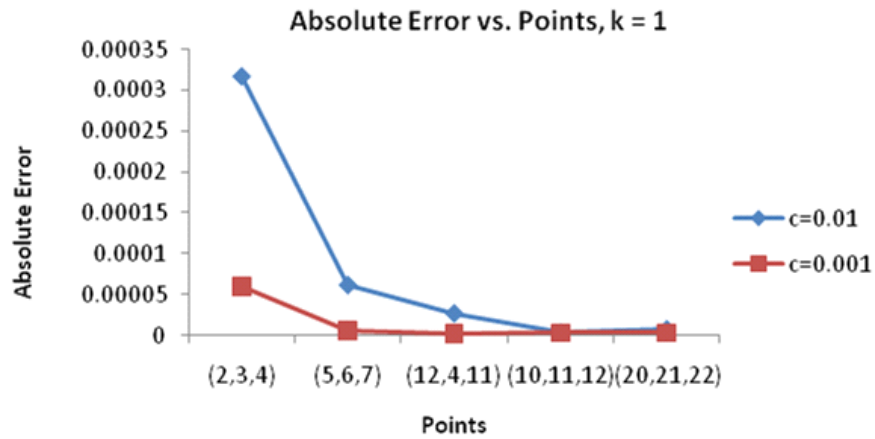
Figure 4: Table 1 and Table 2's data is reflected graphically below.

TABLE 3 $k = 5, c = 0.001, NDEG = 7, NINTI = 32, NINTE = 16$, true solution u_1

| <i>points</i> | <i>absolute error</i> |
|---------------|------------------------|
| (16, 18, 20) | 6.870×10^{-6} |
| (10, 11, 12) | 7.593×10^{-6} |
| (5, 6, 7) | 4.875×10^{-5} |
| (3, 4, 5) | 1.634×10^{-4} |
| (2, 3, 4) | 3.393×10^{-4} |

TABLE 4 $k = 5, c = 0.01, NDEG = 7, NINTI = 32, NINTE = 16$, true solution u_1

| <i>points</i> | <i>absolute error</i> |
|---------------|------------------------|
| (10, 11, 12) | 4.545×10^{-5} |
| (16, 18, 20) | 1.725×10^{-4} |
| (5, 6, 7) | 3.680×10^{-4} |
| (2, 3, 4) | 5.912×10^{-4} |
| (3, 4, 5) | 1.124×10^{-3} |

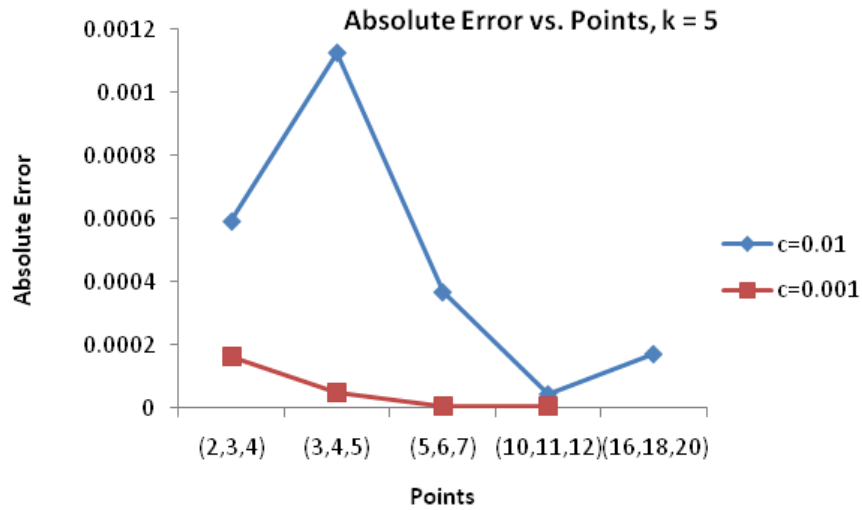
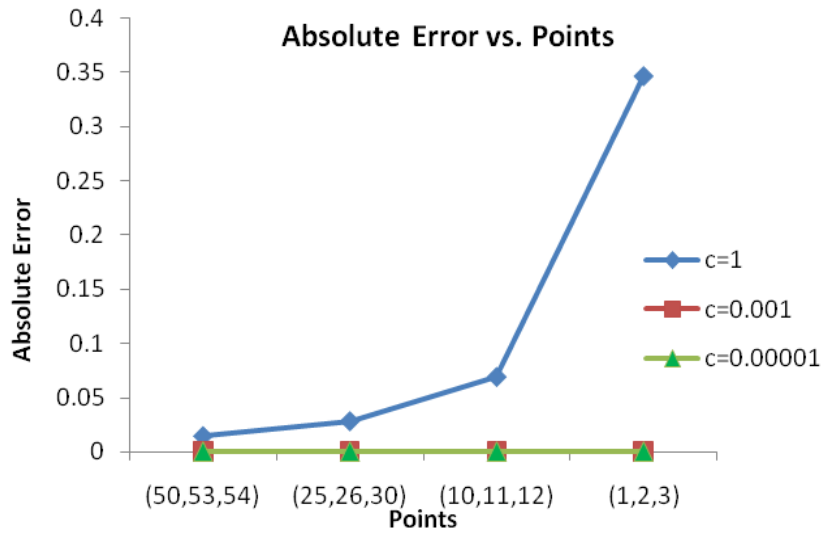
Figure 5: Table 3 and Table 4's data is reflected graphically below.

TABLE 5 $k = 1, NDEG = 7, NINTI = 8, NINTE = 16$, true solution u_1

| <i>points</i> | <i>absolute error when $c = 1$</i> | <i>absolute error when $c = 0.001$</i> | <i>absolute error when $c = 0.000001$</i> |
|---------------|---|---|--|
| (50, 53, 54) | 0.015 | 2.26×10^{-5} | 4.09×10^{-7} |
| (25, 26, 30) | 0.028 | 4.26×10^{-5} | 6.41×10^{-7} |
| (10, 11, 12) | 0.069 | 1.06×10^{-4} | 1.79×10^{-6} |
| (5, 6, 7) | 0.150 | 1.91×10^{-4} | 3.22×10^{-6} |
| (1, 2, 3) | 0.346 | 5.73×10^{-4} | 9.10×10^{-6} |

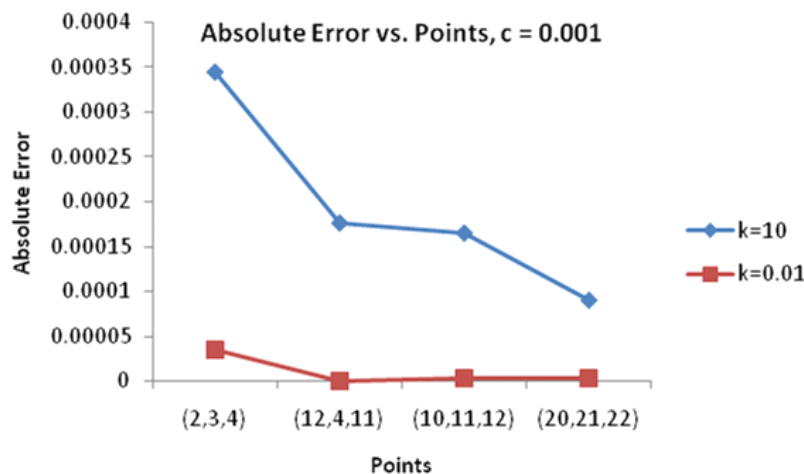
Figure 6: Table 5's data is reflected graphically below.**TABLE 6** $k = 10, c = 0.001, NDEG = 7, NINTI = 32, NINTE = 16$, true solution u_1

| <i>points</i> | <i>absolute error</i> |
|---------------|------------------------|
| (20, 21, 22) | 9.057×10^{-5} |
| (10, 11, 12) | 1.653×10^{-4} |
| (12, 4, 11) | 1.765×10^{-4} |
| (2, 3, 4) | 3.444×10^{-4} |

TABLE 7 $k = 0.01, c = 0.001, NDEG = 7, NINTI = 32, NINTE = 16$, true solution u_1

| <i>points</i> | <i>absolute error</i> |
|---------------|------------------------|
| (12, 4, 11) | 1.696×10^{-7} |
| (10, 11, 12) | 2.855×10^{-6} |
| (20, 21, 22) | 2.632×10^{-6} |
| (2, 3, 4) | 3.445×10^{-5} |

Figure 7: Table 6 and Table 7's data is reflected graphically below.



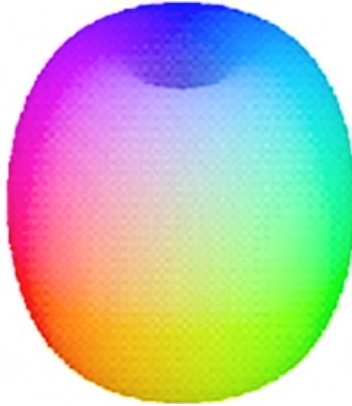
Most recently, we have explored increasing the wave number, k , to 10 and also decreasing k to 0.01. As seen in Table 6 and 7, in conjunction with a low constant value, c , and increased nodes, we arrived at good convergence rates.

Conclusion and Further Remarks

The pseudosphere is a hyperbolic region for which we are altering its coefficient, c , values of the z coordinate making the surface spherical. This causes the shape to become smoother and allow waves to reflect better and arrive at better convergence rates. One of the most significant differences in convergence errors arrived once we altered both the c values and the wave numbers, k . When c was changed from a hyperbolic value to a spherical one, we got better convergence rates. This happened when c was small, such as 0.0001. Similarly, when k equaled values like 10, 5, or 0.01, either stronger or weaker waves hitting the pseudospheres' surface, the convergence errors were also low. Lastly, when we added more interior nodes (referred to as NINTI) and exterior nodes (referred to as NINTE), we got better convergence errors. Furthermore, we always used more interior nodes than exterior because the interior of the shape is smoother, which lends better convergence values. We can use any of these qualities to arrive at better errors; more nodes in conjunction with a more spherical constant and a stronger wave number would give better results.

The torus is shape that can be explored for further research. Unmodified it resembles an inner tube, however, using Maple, we obtain a shape that is pictured below.

Figure 8: The torus.



The equation for the torus is as follows,

$$x = (1 + \cos u)(\cos v), y = (1 + \cos u)(\sin v), z = (\sin u). \quad (15)$$

The radius of the torus is defined as $r = \sqrt{x^2 + y^2 + z^2}$. Applied to the original equation, $r = \sqrt{2}\sqrt{1 + \cos u}$.

Acknowledgements

This thesis would not have been possible without the extensive help from Dr. Yajni Warnapala, Dr. Scott Rutherford, Dr. Annela Kelly and my fellow student, Raveena Siegel. I would also like to thank Dr. Yalor and Jim Levine, for their efforts. Without Mrs. Kennedy I would have never been able to make Scientific Word work, or make my end product look as lovely. Lastly, without the generous grant from the Provost Fund, our trip to present at the AMS Conference for Numerical Analysis would not have been possible. Thank you all very much.

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Appendix A: Fortran Subroutine

As discussed prior, the program entitled Fortran 77 is used to solve the boundary problems for this project. Below is a description of the surface subroutine for the pseudosphere.

THIS IS A PSEUDOSPHERICAL REGION.

```
X=A*CSP*SNT
Y=B*SNP*SNT
Z=C*CST+A*LN(SQRT((1-CST)/(1+CST)))
IF (IFLAG .EQ. 0) RETURN
DX(1)=-A*SNT*SNP
DX(2)=A*CSP*CST
DY(1)=B*CSP*SNT
DY(2)=B*SNP*CST
DZ(1)=ZERO
DZ(2)=-C*SNT+A/SNT
RETURN
```

Appendix B: Example of Using the Galerkin Method

$y''(x) + y(x) + 2x(1 - x) = 0$ with $y(0) = 0$, $y(1) = 0$ is a linear, second order differential equation with the given boundary conditions and basis functions, u_1 , u_2 , u_3 , which are as follows

$$u_0(x) = 0$$

$$u_1(x) = x(x - 1)$$

$$u_2(x) = x^2(x - 1)^2$$

$$u_3(x) = x^3(x - 1)^3.$$

In general, solution of the Galerkin method is $u(x) = u_0(x) + \sum_{i=1}^3 \alpha_i u_i(x)$.

Therefore, writing $u_1(x)$, $u_2(x)$, $u_3(x)$ as a linear combination,

$$u(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x) + \alpha_3 u_3(x)$$

$$u(x) = \alpha_1 x(x - 1) + \alpha_2 x^2(x - 1)^2 + \alpha_3 x^3(x - 1)^3$$

When we substitute $u(x)$ into the original differential equation, we obtain,

$$r(x) = u''(x) + u(x) + 2x(1 - x)$$

and,

$$r(x) = 2x - 2x^2 + (2 - x + x^2)\alpha_1 + (2 - 12x + 13x^2 - 2x^3 + x^4)\alpha_2 + (-6x + 36x^2 - 61x^3 + 33x^4 - 3x^5 + x^6)\alpha_3.$$

Galerkin's method requirement is that the inner product of $r(x)$ with $u_1(x)$, $u_2(x)$, and $u_3(x)$ is zero.

Therefore,

$$\int_0^1 u_1(x)r(x)dx = 0, \int_0^1 u_2(x)r(x)dx = 0, \int_0^1 u_3(x)r(x)dx = 0.$$

We examine $\int_0^1 u_1(x)r(x)dx = 0$ below,

$$\int_0^1 x(x-1)[2x-2x^2+(2-x+x^2)\alpha_1+(2-12x+13x^2-2x^3+x^4)\alpha_2+(-6x+36x^2-61x^3+33x^4-3x^5+x^6)\alpha_3]dx = 0.$$

Simplified, using Maple

$$-\frac{1}{15} - \frac{3}{10}\alpha_1 + \frac{5}{84}\alpha_2 - \frac{4}{315}\alpha_3 = 0.$$

Similarly,

$$u_2(x) = \frac{1}{70} + \frac{5}{84}\alpha_1 - \frac{11}{630}\alpha_2 + \frac{61}{13860}\alpha_3 = 0,$$

$$u_3(x) = -\frac{1}{315} - \frac{4}{315}\alpha_1 + \frac{61}{13860}\alpha_2 - \frac{73}{60060}\alpha_3 = 0.$$

Solve for $\alpha_1, \alpha_2, \alpha_3$

$$\alpha_1 = -\frac{1370}{7397}, \alpha_2 = \frac{50688}{273689}, \alpha_3 = -\frac{132}{21053}.$$

Therefore, the solution to the differential equation is given by,

$$y = -\frac{1370}{7397}x(x-1) + \frac{50688}{273689}x^2(x-1)^2 - \frac{132}{21053}x^3(x-1)^3.$$

Appendix C: Change in Formula

Proposition: Due to the fact that the program cannot code for tangent, we must write tangent in terms of sine and cosine and find the derivative in terms of sine and cosine.

Proof: We start with the function, $y = \ln(\tan(\frac{t}{2}))$.

We apply the chain rule to obtain the derivative, y' .

$$y' = [\frac{1}{\tan(\frac{t}{2})}] (\frac{1}{2}) (\sec(\frac{t}{2}))^2$$

We simplify into one, simpler fraction,

$$y' = \frac{(\sec(\frac{t}{2}))^2}{2(\tan(\frac{t}{2}))}.$$

Now, we change the trigonometric terms into sine functions and cosine functions,

$$y' = \frac{1}{2} [(\frac{1}{(\cos(\frac{t}{2}))^2}) ((\frac{\cos(\frac{t}{2})}{\sin(\frac{t}{2})}))].$$

We cancel terms,

$$y' = \frac{1}{2} [\frac{1}{(\cos(\frac{t}{2}))(\sin(\frac{t}{2}))}].$$

Now, to simplify even further, we apply the trigonometric identity that states $2 \sin(t) \cos(t) = \sin(2t)$, arriving at the final derivative,

$$y' = \frac{1}{\sin(t)} \square.$$

Secondly, we can write the pseudosphere in Cartesian coordinates which is how we must write the equation to obtain graphs in Maple. Since we know the equation of the sphere is given by, $x^2 + y^2 + z^2 = r^2$ and we

have $x = \cos(u) \sin(v)$, $y = \sin(u) \sin(v)$ and $z = \cos(v) + c \ln(\tan(\frac{v}{2}))$, as equations for the pseudosphere, we can substitute into the equation.

This yields:

$$(\cos(u) \sin(v))^2 + (\sin(u) \sin(v))^2 + (\cos(v) + c \ln(\tan(\frac{v}{2})))^2 = r^2.$$

We simplify as follows:

$$\cos^2(u) \sin^2(v) + \sin^2(u) \sin^2(v) + \cos^2(v) + 2c \cos(v) \ln(\tan(\frac{v}{2})) + (c \ln(\tan(\frac{v}{2})))^2 = r^2.$$

Factor out a $\sin^2(v)$:

$$\sin^2(v)(\cos^2(u) + \sin^2(u)) + \cos^2(v) + 2c \cos(v) \ln(\tan(\frac{v}{2})) + (c \ln(\tan(\frac{v}{2})))^2 = r^2$$

Simplify and redistribute:

$$\sin^2(v) + \cos^2(v) + 2c \cos(v) \ln(\tan(\frac{v}{2})) + (c \ln(\tan(\frac{v}{2})))^2 = r^2.$$

Again, apply trigonometric identities:

$$1 + 2c \cos(v) \ln(\tan(\frac{v}{2})) + c^2 (\ln(\tan(\frac{v}{2})))^2 = r^2.$$

Square root of both sides:

$$\sqrt{1 + 2c \cos(v) \ln(\tan(\frac{v}{2})) + c^2 (\ln(\tan(\frac{v}{2})))^2} = r^2.$$

Therefore we can say that the radius of a sphere in Cartesian coordinates, with u and v is given by

$$\sqrt{1 + 2c \cos(v) \ln(\tan(\frac{v}{2})) + c^2 (\ln(\tan(\frac{v}{2})))^2}.$$

Appendix D: Solutions to the Helmholtz Equation

Theorem 1:

If $u = \frac{e^{ikr}}{r}$, then $\Delta u + k^2 u = 0$

Proof 1:

We consider: $r = \sqrt{x^2 + y^2 + z^2}$. We replace into equation $u = \frac{e^{ikr}}{r}$.

Therefore we result with $u = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$.

We take the first and second derivative of u by taking the partial derivatives with respect to x , y and z .

This yields (after simplification): $\frac{-k^2 e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$.

We replace back $\sqrt{x^2 + y^2 + z^2}$ with $r: \frac{-k^2 e^{ikr}}{r}$.

We now substitute u and Δu into the equation and see if it works out:

$$\frac{-k^2 e^{ikr}}{r} + \frac{k^2 e^{ikr}}{r} = 0.$$

Indeed it checks out.

Theorem 2:

If $u = \frac{e^{ikr}}{r}(1 + \frac{i}{kr})z$, then $\Delta u + k^2 u = 0$

Proof 2:

We consider: $r = \sqrt{x^2 + y^2 + z^2}$. We replace into equation $u = \frac{e^{ikr}}{r}(1 + \frac{i}{kr})z$.

Therefore we result with $u = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}(1 + \frac{i}{k\sqrt{x^2+y^2+z^2}})z$.

We take the first and second derivative of u by taking the partial derivatives with respect to x , y and z .

This yields (after simplification): $\frac{-k^2 e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}(1 + \frac{i}{k\sqrt{x^2+y^2+z^2}})z$.

We replace $\sqrt{x^2+y^2+z^2}$ back with r : $\frac{-k^2 e^{ikr}}{r^2}(1 + \frac{i}{kr})z$.

We now substitute u and Δu into the equation and see if it works out:

$$\frac{-k^2 e^{ikr}}{r^2}(1 + \frac{i}{kr})z + \frac{k^2 e^{ikr}}{r^2}(1 + \frac{i}{kr})z = 0$$

Indeed it checks out.

Theorem 3:

If $u = \frac{e^{ikr}}{r^3}(-1 + \frac{3}{k^2 r^2} - \frac{3i}{kr})(0.5)(3z^2 - r^2)$, then $\Delta u + k^2 u = 0$

Proof 3:

We consider: $r = \sqrt{x^2 + y^2 + z^2}$. We replace into equation $u = \frac{e^{ikr}}{r^3}(-1 + \frac{3}{k^2 r^2} - \frac{3i}{kr})(0.5)(3z^2 - r^2)$.

Therefore we result with $u = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{\frac{3}{2}}}(-1 + \frac{3}{k^2(x^2+y^2+z^2)} - \frac{3i}{k\sqrt{x^2+y^2+z^2}})(0.5)(3z^2 - (x^2 + y^2 + z^2))$.

We take the first and second derivative of u by taking the partial derivatives with respect to x , y and z .

This yields (after simplification): $u = \frac{-k^2 e^{ik\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{\frac{3}{2}}}(-1 + \frac{3}{k^2(x^2+y^2+z^2)} - \frac{3i}{k\sqrt{x^2+y^2+z^2}})(0.5)(3z^2 - (x^2 + y^2 + z^2))$.

We replace back $\sqrt{x^2 + y^2 + z^2}$ with r : $\frac{-k^2 e^{ikr}}{r^3}(-1 + \frac{3}{k^2 r^2} - \frac{3i}{kr})(0.5)(3z^2 - r^2)$.

We now substitute u and Δu into the equation and see if it works out:

$$\frac{-k^2 e^{ikr}}{r^3}(-1 + \frac{3}{k^2 r^2} - \frac{3i}{kr})(0.5)(3z^2 - r^2) + \frac{k^2 e^{ikr}}{r^3}(-1 + \frac{3}{k^2 r^2} - \frac{3i}{kr})(0.5)(3z^2 - r^2) = 0$$

Indeed it checks out.

Appendix E: Surface Area of the Pseudosphere

Surface area of the pseudosphere = $2\alpha^2 \int_0^{2\pi} dv \int_0^\infty \tanh(u) \sec h(u) du$

We wish to prove the given equation is equal to $4\pi\alpha^2$, $\alpha =$ the radius, r .

First, we change infinity to a limit of b ,

$$= 2\alpha^2 \int_0^{2\pi} dv \int_0^b \tanh(u) \sec h(u) du$$

Integrate in terms of u variables,

$$\begin{aligned} &= 2\alpha^2 \int_0^{2\pi} dv [\lim_{b \rightarrow \infty} - \sec h(u)] \Big|_0^b \\ &= 2\alpha^2 \int_0^{2\pi} dv [\lim_{b \rightarrow \infty} - \sec h(b) + \sec h(0)] \\ &= 2\alpha^2 \int_0^{2\pi} dv [\lim_{b \rightarrow \infty} - \frac{1}{\cosh(b)} + \frac{1}{\cosh(0)}] \end{aligned}$$

$$= 2\alpha^2 \int_0^{2\pi} dv[0 + 1]$$

$$= 2\alpha^2 \int_0^{2\pi} dv[1]$$

Evaluate,

$$= 2\alpha^2 \int_0^{2\pi} dv$$

Integrate in terms of v variables,

$$= 2\alpha^2 [v]_0^{2\pi}$$

Evaluate,

$$= 2\alpha^2(2\pi)$$

Simplify,

$$= 4\pi\alpha^2 \square.$$